Integrable polynomial factorization for symplectic systems

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It has been shown that an analytic symplectic map can be directly converted into a product of Lie transformations in the form of integrable polynomial factorization with the desired accuracy. A map in the form of integrable polynomial factorization is exactly symplectic and easy to evaluate exactly. Error involved in the integrable polynomial factorization has been studied with the case of the Hénon map. The results suggest that the map in the form of integrable polynomial factorization is a reliable and convenient model for the study of the long-term behavior of a symplectic system such as a large storage ring.

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I. INTRODUCTION

The study of Hamiltonian systems can often be accomplished by using one-turn maps, maps on the Poincaré sections—a great conceptual and computational simplification. For a complicated, system such as a large storage ring with thousands of nonlinear elements, however, it is impractical to extract an exact one-turn map even though analytic forms of Hamiltonians for individual elements are known. Instead, a truncated Taylor expansion of the one-turn map can be easily obtained through concentrating the actions of individual elements by means of Lie and differential algebra [1–3]. The truncation inevitably violates the symplectic condition and, as a consequence, leads to spurious effects if the truncated map is used to study the long-term behavior of the system [3]. One way to recover the symplecticity is to convert the truncated map order by order into a product of Lie transformations by means of the Dragt-Finn factorization [4]. A map in the form of Lie transformations is guaranteed to be symplectic, but it generally cannot be used for tracking directly because evaluating a nonlinear map in such a form is equivalent to solving nonlinear Hamiltonian systems, which cannot be done in general. In a previous work [5], we have found that Lie transformations associated with a special class of homogeneous polynomials, called integrable polynomials, can be evaluated exactly. Moreover, any homogeneous polynomial can be written as a sum of integrable polynomials of the same degree. A symmetric integrable polynomial factorization has thus been developed to convert a symplectic map in the form of Dragt-Finn factorization into a product of Lie transformations associated with integrable polynomials by using symplectic integrators [5]. A map in the form of integrable polynomial factorization, called an integrable polynomial factorization map, is exactly symplectic and easy to evaluate exactly. In this paper we will show that an integrable polynomial factorization map can be directly obtained from a symplectic system without invoking Dragt-Finn factorization. With this scheme, the total number of nonlinear kicks in the integrable polynomial factorization map is greatly reduced. Furthermore, the elimination of unnecessary computation for Dragt-Finn factorization makes the integrable polynomial factorization more accurate.

The paper is organized as follows: In Sec. II we describe the concept of a truncated symplectic map and briefly recall some of the results of the integrable polynomial for Lie transformations. The integrable polynomial factorization is discussed in Sec. III. In Sec. IV the error involved in the integrable polynomial factorization is studied with an example of the Hénon map. Section V contains our conclusions

II. THE TAYLOR MAP OF SYMPLECTIC SYSTEMS AND INTEGRABLE POLYNOMIALS FOR LIE TRANSFORMATIONS

At any "checkpoint" of an accelerator, motions of particles can be described mathematically by a six-dimensional symplectic one-turn map,

$$\mathbf{z}'=\mathcal{M}\mathbf{z}\ , \tag{1}$$

where

$$\mathbf{z}=(z_1,z_2,z_3,z_4,z_5,z_6)$$

is a phase-space vector and z_{i+3} is the conjugate momenta of z_i for $i=1$, 2, and 3. In this paper we shall be working with this six-dimensional phase space. The results can, however, be applied to systems with any degree of freedom. $\mathcal{M}$ is, in general, a nonlinear functional operator. Because we are usually not interested in transformations that simply translate the origin in phase space, only maps that map the origin to itself ($\mathbf{z}=0$ is the closed orbit) are considered. Within its analytic domain, $\mathcal{M}$ can be expanded in a power series of $\mathbf{z}$,

$$\mathbf{z}'=\sum_{i=1}^{\infty}\mathbf{U}(i,\mathbf{z})=\sum_{i=1}^{N}\mathbf{U}(i,\mathbf{z})+\epsilon(N+1)\ , \tag{2}$$

where $\mathbf{U}(i,\mathbf{z})$ is a vectorial homogeneous polynomial in $\mathbf{z}$ of degree i,

$$\mathbf{U}(i,\mathbf{z})=\sum_{|\sigma|=i}\mathbf{u}(\sigma)z_1^{\sigma_1}z_2^{\sigma_2}z_3^{\sigma_3}z_4^{\sigma_4}z_5^{\sigma_5}z_6^{\sigma_6}\ , \tag{3}$$

and $\epsilon(N+1)$ represents a remainder series consisting of terms higher than the Nth order. In Eq. (3), $\mathbf{u}(\sigma)$ are constant coefficients, σ denotes a collection of exponents $(\sigma_1,\ldots,\sigma_6)$, and $|\sigma|=\sum_{j=1}^{6}\sigma_j$. Truncating the expansion in Eq. (2) at the Nth order results in an Nth-order Taylor map. Since such truncation inevitably violates the symplectic condition, the Taylor map typically produces spurious damping or growth when used to study the long-term behavior of trajectories. In order to use a one-turn map to study the long-term stability, the Taylor map has thus to be replaced by a symplectic map that can be easy to evaluate exactly and whose effect is identical to that of the Taylor map through some order. It will be shown in the following that such a symplectic map can be constructed as a product of Lie transformations in the form of integrable polynomial factorization with an accuracy up to the truncated order of the Taylor map.

As a prelude to Sec. III, we briefly recall some of the definitions and results of the integrable polynomial for Lie transformations. The details can be found in Ref. [5]. For any analytic function $g(\mathbf{z})$, a Lie transformation can be defined by the exponential series

$$\exp(:g:)=\sum_{n=0}^{\infty}\frac{1}{n!}(:g:)^n , \tag{4}$$

where $:g:$ denotes the Lie operator associated with g, which is defined by the Poisson bracket operation

$$:g:=\sum_{i=1}^{3}\left[\frac{\partial g}{\partial z_i}\frac{\partial}{\partial z_{i+3}}-\frac{\partial g}{\partial z_{i+3}}\frac{\partial}{\partial z_i}\right] . \tag{5}$$

A polynomial $g(\mathbf{z})$ is called an integrable polynomial if its associated Lie transformation can be evaluated exactly, i.e., $\exp[:g(\mathbf{z}):]\mathbf{z}$, can be expressed as an explicit function of $\mathbf{z}$. Let

$$\{g_i^{(k)}(\mathbf{z})\,|\,k=1,2,\ldots,N_g(i)\}$$

denote a complete set of integrable polynomials of degree i, where $N_g(i)$ is the number of integrable polynomials of degree i. It was shown [5] that any polynomial of $\mathbf{z}$ can be written as a sum of integrable polynomials of the same degree, i.e.,

$$\begin{aligned} f_i(\mathbf{z}) &= \sum_{|\sigma|=i} a(\sigma) z_1^{\sigma_1} z_2^{\sigma_2} z_3^{\sigma_3} z_4^{\sigma_4} z_5^{\sigma_5} z_6^{\sigma_6} \\ &= \sum_{k=1}^{N_g(i)} g_i^{(k)}(\mathbf{z}) , \end{aligned} \tag{6}$$

where $f_i(\mathbf{z})$ is any homogeneous polynomial of degree i and $a(\sigma)$ are constant coefficients. In the six-dimensional phase space, $N_g(i)$ is 8, 20, 42, and 79 for $i=3$, 4, 5, and 6, respectively [5]. Reference [5] contains the details of the method of constructing integrable polynomials and a list of integrable polynomials and their associated Lie transformations expressed as explicit functions of $\mathbf{z}$ for degree 3 to 6.

III. INTEGRABLE POLYNOMIAL FACTORIZATION

Theorem 1 (factorization theorem). A symplectic map in Eq. (2) can be written as a product of Lie transformations in the form of integrable polynomial factorization with an accuracy up to the truncated order N, i.e.,

$$\mathbf{z}'=\mathcal{R}\prod_{i=3}^{N+1}\left[\prod_{k=1}^{N_g(i)}\exp(:g_i^{(k)}:)\right]\mathbf{z}+\sigma(N+1) , \tag{7}$$

where $\mathcal{R}$ denotes a linear symplectic transformation.

The symplectic map defined by Eq. (7) is called an integrable polynomial factorization map. Since each $\exp(:g_i^{(k)}:)\mathbf{z}$ can be expressed as an explicit function of $\mathbf{z}$, an integrable polynomial factorization map is composed of simple iterations (kicks) and is easy to evaluate exactly. The number of iterations of the Nth-order map is $1+\sum_{i=3}^{N+1}N_g(i)$. For example, the fifth-order integrable polynomial factorization map in the six-dimensional phase space consists of 150 iterations. In the following, we shall prove theorem 1 by showing that all coefficients of integrable polynomials $g_i^{(k)}$ can be obtained order by order from the coefficients of the Taylor map. The procedure is similar to the proof of the factorization theorem for Dragt-Finn factorization [1].

We first demonstrate that $\mathcal{R}$ is symplectic and corresponds to the linear Taylor map. Let M be the Jacobian matrix of $\mathcal{M}$ at $\mathbf{z}=0$, which can be computed from Eqs. (2) and (7) as

$$M\mathbf{z}=\mathcal{R}\mathbf{z}=\mathbf{U}(1,\mathbf{z}) . \tag{8}$$

Since $\mathcal{M}$ is symplectic, M is a symplectic matrix and hence $\mathcal{R}$ is a linear symplectic transformation associated with M. Multiplying $\mathcal{R}^{-1}$ on both sides of Eq. (2) yields

$$\begin{aligned} \mathcal{R}^{-1}\mathbf{z}' &= \mathbf{z}+\sum_{i=2}^{\infty}\mathbf{U}^{(1)}(i,\mathbf{z}) \\ &= \mathbf{z}+\epsilon(2) , \end{aligned} \tag{9}$$

where

$$\mathbf{U}^{(1)}(i,\mathbf{z})=\mathcal{R}^{-1}\mathbf{U}(i,\mathbf{z})=\mathbf{U}(i,M^{-1}\mathbf{z}) \tag{10}$$

is a vectorial homogeneous polynomial in $\mathbf{z}$ of degree i.

Next we will show that $g_i^{(k)}$ in Eq. (7) can be obtained order by order by applying

$$\prod_{k=N_g(i)}^{1}\exp(:-g_i^{(k)}:)$$

to both sides of Eq. (9) successively. In order to simplify the discussion, we need to have two preparations. First, it can be easily shown by using Eq. (4) that for any function $f(\mathbf{z})$,

$$\prod_{k=N_g(i)}^{1}\exp(:-g_i^{(k)}:)f(\mathbf{z})=f(\mathbf{z})+\sum_{n=1}^{\infty}\mathcal{G}_{in}f(\mathbf{z}), \tag{11}$$

where

$$\mathcal{G}_{i1}=\sum_{k=1}^{N_g(i)} :-g_i^{(k)}: , \tag{12}$$

$$\mathcal{G}_{i2}=\frac{1}{2!}\sum_{k=1}^{N_g(i)}(:-g_i^{(k)}:)^2+\sum_{k_1=2}^{N_g(i)}\sum_{k_2=1}^{k_1-1} :-g_i^{(k_1)}::-g_i^{(k_2)}: , \tag{13}$$

$$\mathcal{G}_{i3}=\frac{1}{3!}\sum_{k=1}^{N_g(i)}(:-g_i^{(k)}:)^3+\frac{1}{2!}\sum_{k_1=2}^{N_g(i)}\sum_{k_2=1}^{k_1-1}\sum_{\mathbf{m}=P(1,2)}(:-g_i^{(k_1)}:)^{m_1}(:-g_i^{(k_2)}:)^{m_2}$$
$$+\sum_{k_1=3}^{N_g(i)}\sum_{k_2=2}^{k_1-1}\sum_{k_3=1}^{k_2-1} :-g_i^{(k_1)}::-g_i^{(k_2)}::-g_i^{(k_3)}: , \tag{14}$$

$$\mathcal{G}_{i4}=\frac{1}{4!}\sum_{k=1}^{N_g(i)}(:-g_i^{(k)}:)^4+\frac{1}{3!}\sum_{k_1=2}^{N_g(i)}\sum_{k_2=1}^{k_1-1}\sum_{\mathbf{m}=P(1,3)}(:-g_i^{(k_1)}:)^{m_1}(:-g_i^{(k_2)}:)^{m_2}$$
$$+\frac{1}{2!2!}\sum_{k_1=2}^{N_g(i)}\sum_{k_2=1}^{k_1-1}(:-g_i^{(k_1)}:)^2(:-g_i^{(k_2)}:)^2$$
$$+\frac{1}{2!}\sum_{k_1=3}^{N_g(i)}\sum_{k_2=2}^{k_1-1}\sum_{k_3=1}^{k_2-1}\sum_{\mathbf{m}=P(1,1,2)}(:-g_i^{(k_1)}:)^{m_1}(:-g_i^{(k_2)}:)^{m_2}(:-g_i^{(k_3)}:)^{m_3}$$
$$+\sum_{k_1=4}^{N_g(i)}\sum_{k_2=3}^{k_1-1}\sum_{k_3=2}^{k_2-1}\sum_{k_4=1}^{k_3-1} :-g_i^{(k_1)}::-g_i^{(k_2)}::-g_i^{(k_3)}::-g_i^{(k_4)}:$$
$$\vdots \; . \tag{15}$$

In Eqs. (14) and (15), $\sum_{\mathbf{m}=P(l_1,\ldots,l_n)}$ denotes the sum of all different permutations of n integers $(l_1,\ldots,l_n)$ for n exponents $\mathbf{m}=(m_1,\ldots,m_n)$. For example,

$$\sum_{\mathbf{m}=P(1,2)}(:-g_i^{(k_1)}:)^{m_1}(:-g_i^{(k_2)}:)^{m_2}=:-g_i^{(k_1)}:(:-g_i^{(k_2)}:)^2+(:-g_i^{(k_1)}:)^2:-g_i^{(k_2)}: . \tag{16}$$

The action of $\mathcal{G}_{in}$ on any homogeneous polynomial of degree j results in a homogeneous polynomial of degree $(i-2)n+j$. Second, we will use the following lemma without giving the proof. The proof can be found in Ref. [1].

Lemma 1. Let $\mathbf{U}(i,\mathbf{z})$ be a vectorial homogeneous polynomial in $\mathbf{z}$ of degree i. Suppose it satisfies the relations

$$[z_l,U_j(\mathbf{z})]=[z_j,U_l(\mathbf{z})] , \tag{17}$$

where U_j and U_l are the jth and lth components of $\mathbf{U}(i,\mathbf{z})$, respectively. Then such a polynomial exists if and only if there is a homogeneous polynomial $g_{i+1}(\mathbf{z})$ of degree $i+1$ such that

$$\mathbf{U}(\mathbf{z})=[g_{i+1},\mathbf{z}] . \tag{18}$$

Taking the Poisson bracket of both sides of Eq. (9) with itself for different components yields

$$0=[z_l,U_j^{(1)}(2,\mathbf{z})]+[U_l^{(1)}(2,\mathbf{z}),z_j]+\epsilon(2) , \tag{19}$$

where $U_j^{(1)}$ and $U_l^{(1)}$ are the jth and lth components of $\mathbf{U}^{(1)}$, respectively. Equating terms of like degree in Eq. (19) gives the result

$$[z_l,U_j^{(1)}(2,\mathbf{z})]+[U_l^{(1)}(2,\mathbf{z}),z_j]=0 , \tag{20}$$

that is, $\mathbf{U}^{(1)}(2,\mathbf{z})$ in Eq. (9) satisfies the condition in Eq. (17) of lemma 1.

Now apply $\prod_{k=8}^{1}\exp(:-g_3^{(k)}:)$ to both sides of Eq. (9). By using Eq. (11), one obtains

$$\prod_{n=8}^{1}\exp(:-g_3^{(n)}:)\mathcal{R}^{-1}\mathbf{z}'$$
$$=\mathbf{z}+\sum_{k=1}^{8} :-g_3^{(k)}:\mathbf{z}+\mathbf{U}^{(1)}(2,\mathbf{z})$$
$$+\sum_{i=3}^{\infty}\mathbf{U}^{(1)}(i,\mathbf{z})+\mathcal{G}_{31}\sum_{i=2}^{\infty}\mathbf{U}^{(1)}(i,\mathbf{z})$$
$$+\sum_{i=2}^{\infty}\mathcal{G}_{3i}\left[\mathbf{z}+\sum_{i=2}^{\infty}\mathbf{U}^{(1)}(i,\mathbf{z})\right] . \tag{21}$$

According to lemma 1, there exists a homogeneous polynomial $\sum_{k=1}^{8}g_3^{(k)}(\mathbf{z})$ of degree 3 such that

$$\sum_{k=1}^{8} :-g_3^{(k)}:\mathbf{z}+\mathbf{U}^{(1)}(2,\mathbf{z})=0 , \tag{22}$$

and Eq. (21) then becomes

$$\prod_{k=8}^{1} \exp(:-g_3^{(k)}:)\mathcal{R}^{-1}\mathbf{z}'=\mathbf{z}+\sum_{i=3}^{\infty}\mathbf{U}^{(2)}(i,\mathbf{z}) =\mathbf{z}+\epsilon(3) , \tag{23}$$

where $\mathbf{U}^{(2)}(i,\mathbf{z})$ are homogeneous polynomials in $\mathbf{z}$ of degree i which are defined by

$$\mathbf{U}^{(2)}(3,\mathbf{z})=\mathbf{U}^{(1)}(3,\mathbf{z})+\mathcal{G}_{32}\mathbf{z}+\mathcal{G}_{31}\mathbf{U}^{(1)}(2,\mathbf{z}) , \tag{24}$$

$$\mathbf{U}^{(2)}(4,\mathbf{z})=\mathbf{U}^{(1)}(4,\mathbf{z})+\mathcal{G}_{33}\mathbf{z}+\mathcal{G}_{32}\mathbf{U}^{(1)}(2,\mathbf{z}) +\mathcal{G}_{31}\mathbf{U}^{(1)}(3,\mathbf{z}) , \tag{25}$$

$$\mathbf{U}^{(2)}(5,\mathbf{z})=\mathbf{U}^{(1)}(5,\mathbf{z})+\mathcal{G}_{34}\mathbf{z}+\mathcal{G}_{33}\mathbf{U}^{(1)}(2,\mathbf{z}) +\mathcal{G}_{32}\mathbf{U}^{(1)}(3,\mathbf{z})+\mathcal{G}_{31}\mathbf{U}^{(1)}(4,\mathbf{z}) \; \vdots . \tag{26}$$

$\sum_{k=1}^{8} g_3^{(k)}(\mathbf{z})$ should be solved from Eq. (22). Because of Eq. (20),

$$\sum_{i=1}^{3}\left[\frac{\partial}{\partial z_i}\left[\sum_{k=1}^{8} g_3^{(k)}\right]dz_i+\frac{\partial}{\partial z_{i+3}}\left[\sum_{k=1}^{8} g_3^{(k)}\right]_{i+3}\right]$$

is an exact differential and Eq. (22) can be solved as

$$\sum_{k=1}^{8} g_3^{(k)}=-\int_0^{\mathbf{z}}\sum_{i=1}^{3}[U_{i+3}^{(1)}(2,\bar{\mathbf{z}})d\bar{z}_i-U_i^{(1)}(2,\bar{\mathbf{z}})d\bar{z}_{i+3}] , \tag{27}$$

where

$$\bar{\mathbf{z}}=(\bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4,\bar{z}_5,\bar{z}_6) .$$

Furthermore, $\sum_{k=1}^{8} g_3^{(k)}$ is independent of the integral path in Eq. (27) so that one can choose a special integral path $\bar{\mathbf{z}}=\lambda\mathbf{z}$ and obtain

$$\sum_{k=1}^{8} g_3^{(k)}=-\tfrac{1}{3}\sum_{i=1}^{3}[z_iU_{i+3}^{(1)}(2,\mathbf{z})-z_{i+3}U_i^{(1)}(2,\mathbf{z})] . \tag{28}$$

The coefficients for integrable polynomial of degree 3 are thus determined by equating like terms on both sides of Eq. (28).

Comparisons of the right-hand sides of Eqs. (9) and (23) shows that the degree of the remainder has been raised by 1. Following the same procedure, one can successively apply

$$\prod_{n=N_g(i)}^{1} \exp(:-g_i^{(n)}:) ,$$

with $i=4,5,\ldots,N+1$ to increase the degree of the remainder to $N+1$ and obtain

$$\prod_{n=N+1}^{1}\left[\prod_{k=N_g(n)}^{1}\exp(:-g_n^{(k)}:)\right]\mathcal{R}^{-1}\mathbf{z}'=\mathbf{z}+\epsilon(N+1) . \tag{29}$$

The integrable polynomials

$$\{g_n^{(k)}|k=1,2,\ldots,N_g(n)\}$$

of degree $n=4,\ldots,N+1$ are given by

$$\sum_{k=1}^{N_g(n)} g_n^{(k)}=-\frac{1}{n}\sum_{l=1}^{3}[z_lU_{l+3}^{(n-2)}(n-1,\mathbf{z}) -z_{l+3}U_l^{(n-2)}(n-1,\mathbf{z})] , \tag{30}$$

where $U_m^{(n-2)}(n-1,\mathbf{z})$ is the mth component of homogeneous polynomial $\mathbf{U}^{(n-2)}(n-1,\mathbf{z})$ of degree $n-1$, which can be computed iteratively from

$$\mathbf{U}^{(n)}(i,\mathbf{z})=\mathbf{U}^{(n-1)}(i,\mathbf{z})+\sum_{k\ge 2}\delta_{i,(n-2)k+1}\mathcal{G}_{nk}\mathbf{z} +\sum_{k\ge 1}\sum_{l\ge n-1}\delta_{i,(n-2)k+l}\mathcal{G}_{nk}\mathbf{U}^{(n-1)}(l,\mathbf{z}) . \tag{31}$$

In Eq. (31), $n\ge 2$, $i\ge n$, and δ_{mn} is the Kronecker δ function. Inverting the left-hand side of Eq. (29) and neglecting the remainder term, one obtains an integrable polynomial factorization map,

$$\mathbf{z}'=\mathcal{R}\prod_{n=3}^{N+1}\left[\prod_{k=1}^{N_g(n)}\exp(:g_n^{(k)}:)\right]\mathbf{z} . \tag{32}$$

If the remainder term $\epsilon(N+1)$ tends to zero as $N\to\infty$, the symplectic map in Eq. (32) can be used to model the original system with the desired accuracy. In Sec. IV we shall study, with an example, the total error involved in the integrable polynomial factorization map and examine the deviation of the phase-space structure of the map from that of the original system.

IV. ACCURACY OF INTEGRABLE POLYNOMIAL FACTORIZATION

In this section we will apply the result of Sec. III to an exact symplectic map. The integrable polynomial factorization maps will be constructed as approximations to the exact map. The accuracy of the approximations will be studied through comparisons of the exact map and the approximate maps.

Consider a ring with one sextuple kick which is otherwise linear. The betatron motion in the horizontal plane can be described as a composition of linear rotation and a sextuple kick. Let (ξ,η) denote normalized canonical variables. The exact one-turn map is

$$\begin{aligned}\xi'&=c\xi+s(\eta-b\xi^2) ,\\ \eta'&=-s\xi+c(\eta-b\xi^2) ,\end{aligned} \tag{33}$$

where $c=\cos(2\pi\nu)$, $s=\sin(2\pi\nu)$, ν is the linear tune, and b the strength of the sextuple kick. Let $\mathbf{z}=(x,p)=(b\xi,b\eta)$. Map (33) is then transformed into the Hénon map and can be written in the form of

$$\mathbf{z}'=\mathcal{M}\mathbf{z}=\mathbf{U}(1,\mathbf{z})+\mathbf{U}(2,\mathbf{z}) , \tag{34}$$

where

$$\mathbf{U}(i,\mathbf{z})=[U_x(i,\mathbf{z}),U_p(i,\mathbf{z})]$$

and

$$U_x(1,\mathbf{z})=cx+sp\ ,\quad U_p(1,\mathbf{z})=-sx+cp\ , \tag{35}$$

$$U_x(2,\mathbf{z})=-sx^2\ ,\quad U_p(2,\mathbf{z})=-cx^2\ . \tag{36}$$

Figure 1 displays the phase-space portrait of map (34) for $\nu=0.2114$. The dynamic aperture, which is the boundary for global stable motion, is located outside and close to the fifth-order resonance. Note that in map (34) the quadratic terms exceed the linear terms when $x>1$ or $p>1$. The phase-space region of interest is thus

$$x<1 \quad \text{and} \quad p<1\ . \tag{37}$$

We now apply the formalism in Sec. III to construct the integrable polynomial factorization map for map (34). By using Eqs. (10) and (28), we obtain

$$\sum_{k=1}^{8} g_3^{(k)}=-\tfrac{1}{3}c^3x^3+sc^2x^2p-s^2cxp^2+\tfrac{1}{3}s^3p^3\ . \tag{38}$$

Comparison of terms on the right-hand side of Eq. (38) and the standard form of integrable polynomials of degree 3 [5] shows that there are only two nonzero integrable polynomials of degree 3 in Eq. (38), which are

$$g_3^{(1)}=-\tfrac{1}{3}c^2x^3+sc^2x^2p\ , \tag{39}$$

$$g_3^{(2)}=-s^2cxp^2+\tfrac{1}{3}s^3p^3\ . \tag{40}$$

The Lie transformations associated with $g_3^{(1)}$ and $g_3^{(2)}$ are [5]

$$\exp(:g_3^{(1)}:)x=\frac{x}{1+sc^2x}\ , \tag{41}$$

$$\exp(:g_3^{(1)}:)p=\frac{(sc^2p-\frac{1}{3}c^3x)(1+sc^2x)^3+\frac{1}{3}c^3x}{sc^2(1+sc^2x)}\ , \tag{42}$$

$$\exp(:g_3^{(2)}:)x=\frac{(s^2cx-\frac{1}{3}s^3p)(1+s^2cp)^3+\frac{1}{3}s^3p}{s^2c(1+s^2cp)}\ , \tag{43}$$

$$\exp(:g_3^{(2)}:)p=\frac{p}{1+s^2cp}\ . \tag{44}$$

The second-order integrable polynomial factorization map is thus

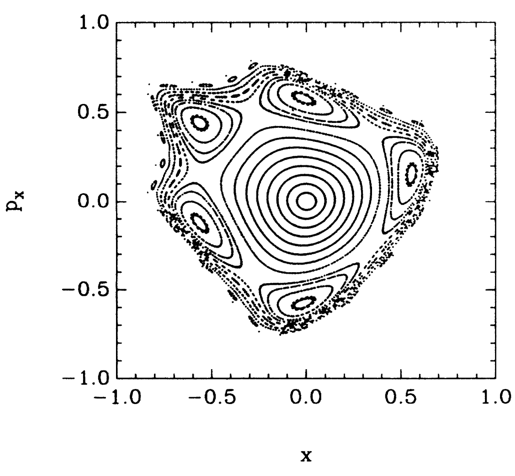

FIG. 1. Phase-space portrait of the Hénon map in Eq. (34) with $\nu=0.2114$.

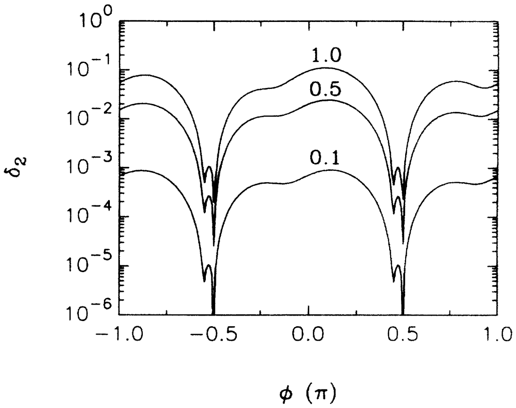

FIG. 2. Accuracy of the second-order integrable polynomial factorization map in Eq. (45) with $\nu=0.2114$. ϕ is the angle variable of polar coordinates. The number on the curve indicates the value of the radius x^2+p^2.

$$\mathbf{z}'=\mathcal{M}_2\mathbf{z}=\mathcal{R}\exp(:g_3^{(1)}:)\exp(:g_3^{(2)}:)\mathbf{z}\ , \tag{45}$$

where $\mathcal{R}$ is the rotation defined by Eq. (35). In order to obtain the next order map, we use Eqs. (24) and (30) and find

$$\sum_{k=1}^{20} g_4^{(k)}=-s^2c^4x^3p+2s^3c^3x^2p^2-s^4c^2xp^3\ . \tag{46}$$

Comparing Eq. (46) to the standard form of integrable polynomials of degree 4 yields

$$g_4^{(9)}=-s^2c^4x^3p\ , \tag{47}$$

$$g_4^{(10)}=2s^3c^3x^2p^2\ , \tag{48}$$

$$g_4^{(11)}=-s^4c^2xp^3\ , \tag{49}$$

and the rest of the integrable polynomials of degree 4 in Eq. (46) are zero. The Lie transformations associated with $g_4^{(9)}$, $g_4^{(10)}$, and $g_4^{(11)}$ are [5]

$$\exp(:g_4^{(9)}:)x=\frac{x}{\sqrt{1-2s^2c^4x^2}}\ , \tag{50}$$

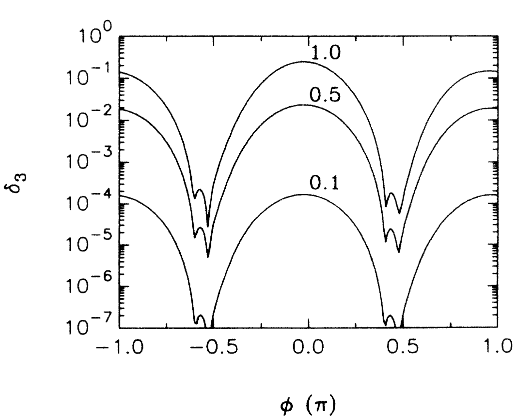

FIG. 3. Accuracy of the third-order integrable polynomial factorization map in Eq. (56) as $\nu=0.2114$. ϕ is the angle variable of polar coordinates. The number on the curve indicates the value of the radius x^2+p^2.

$$\exp(:g_4^{(9)}:)p = p[1-2s^2c^4x^2]^{3/2} , \quad (51)$$

$$\exp(:g_4^{(10)}:)x = x \exp[-4s^3c^3xp] , \quad (52)$$

$$\exp(:g_4^{(10)}:)p = p \exp[4s^3c^3xp] , \quad (53)$$

$$\exp(:g_4^{(11)}:)x = x[1+2s^4c^2p^2]^{3/2} , \quad (54)$$

$$\exp(:g_4^{(11)}:)p = \frac{p}{\sqrt{1+2s^4c^2p^2}} , \quad (55)$$

and the third-order integrable polynomial factorization map is then

$$\mathbf{z}' = \mathcal{M}_3\mathbf{z} = \mathcal{R} \exp(:g_3^{(1)}:)\exp(:g_3^{(2)}:)\exp(:g_4^{(9)}:) \times \exp(:g_4^{(10)}:)\exp(:g_4^{(11)}:)\mathbf{z} . \quad (56)$$

It should be noted that the maps in Eqs. (45) and (56) valid only when $x < 1/|sc^2|$, $p < 1/|s^2c|$ and $1-2s^2c^4x^2>0$, $1+2s^4c^2p^2>0$, respectively. Within the phase-space region of interest defined by Eq. (37), these conditions are always satisfied.

To estimate errors of the approximate maps in Eqs. (45) and (56), we computed [6]

$$\delta_n(\mathbf{z}) = \|\mathcal{M}\mathbf{z} - \mathcal{M}_n\mathbf{z}\| / \|\mathbf{z}\| , \quad (57)$$

where $\|\cdot\|$ denotes the usual Euclidean norm and subscript n denotes the order of integrable polynomial factorization map. In Figs. 2 and 3, $\delta_2(\mathbf{z})$ and $\delta_3(\mathbf{z})$ were plotted, respectively, with polar coordinates defined by $x = \|\mathbf{z}\|\cos\phi$ and $p = \|\mathbf{z}\|\sin\phi$. Inspection of Figs. 2 and 3 shows that the difference between $\mathcal{M}$ and $\mathcal{M}_n$ scales as $\|z\|^{n+1}$. As one increases the truncation order n, this difference can thus be reduced to be as small as desired.

To examine the long-term behavior of the integrable polynomial factorization, we repeatedly apply $\mathcal{M}_2$ and $\mathcal{M}_3$ to various initial conditions, respectively. The results are displayed in Fig. 4 for $\mathcal{M}_2$ and in Fig. 5 for $\mathcal{M}_3$. Comparison of Figs. 1 and 4 shows that within the phase-space region of interest, the long-term behavior of the exact map and the integrable polynomial factorization map are very similar even when the lowest-order approximation was used in the approximate map. Comparison of Figs. 1 and 5 shows that as the order of the approximate map increases, the approximation becomes better. Figure 5 shows that the third-order approximate map can reproduce the 26th-order resonance of the original system in the region where the maps are quite nonlinear.

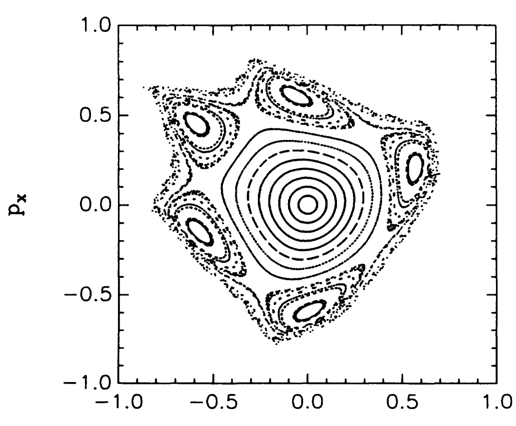

FIG. 4. Phase-space portrait of the second-order integrable polynomial factorization map in Eq. (45) with $\nu = 0.2114$.

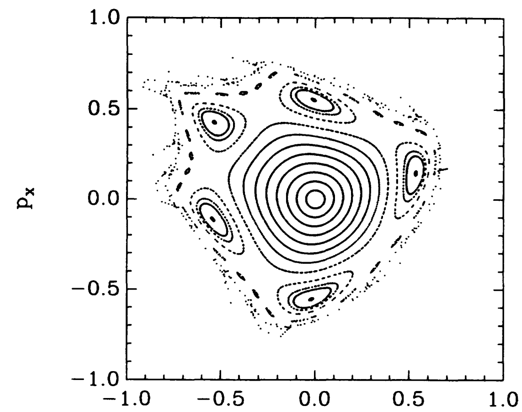

FIG. 5. Phase-space portrait of the third-order integrable polynomial factorization map in Eq. (56) with $\nu = 0.2114$.

It should be noted that even though all data discussed here are from the case of $\nu = 0.2114$, other cases of different linear tune were also examined and the results were found to be similar.

V. CONCLUSIONS

It was shown that an analytic symplectic map can be written as a product of Lie transformations in the form of integrable polynomial factorization. Since the Lie transformation associated with an integrable polynomial can be expressed as an explicit function of phase-space variables, the integrable polynomial factorization map is easy to evaluate exactly. As an example, we have constructed the second- and third-order integrable polynomial factorization maps for the Hénon map. The error involved in the integrable polynomial factorization was studied with the comparisons of the exact and approximate maps. The error was found to be as small as desired. The long-term behavior of integrable polynomial factorization maps were also examined by repeatedly applying the maps to various initial conditions. It was found that the exact and approximate maps have similar long-term behavior even in regions where the maps are quite nonlinear. Over the phase-space region of interest, the original symplectic system can be well approximated by an integrable polynomial factorization map with the desired accuracy and the long-term tracking study can therefore be directly conducted with the integrable polynomial factorization map.

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